An Unrestricted Notion of the Finite Factorization Property

Background

In the integers $\mathbb{Z}$, the **Fundamental Theorem of Arithmetic** guarantees that every number factors uniquely into primes (e.g., $12 = 2^2 \cdot 3$). However, in more general number systems, this may not be the case.

The study of these **factorization properties** is immensely important, powering real-world applications like **RSA encryption** and helping mathematicians tackle the hardest challenges in math—such as in Lamé's attempted proof of **Fermat's Last Theorem**.

Notation

To generalize factorization, we must abstract number systems to their essential parts. Because divisibility relies solely on **multiplication**, we focus on **monoids** (number systems with multiplication only) and **integral domains** (number systems with both addition and multiplication).

Definition. A (multiplicative) **monoid** M is a pair $(M, \times)$, where $\times$ is an associative and commutative binary operation on M and M is a multiplicatively closed set with an identity element.

Atomicity

- An element of M is an **atom** (or irreducible) if its only divisors are units and itself. The set of atoms is $\mathscr{A}(M)$.
- M is **atomic** if every non-zero non-unit element can be written (factorized) as a product of atoms.

Example. $(\mathbb{Z}, \times)$ is an atomic multiplicative monoid with $\mathscr{A}(\mathbb{Z}) = \{2, 3, 5, 7, \dots\}$, the prime numbers. $\mathbb{Z}$ is also an integral domain.

Finiteness Conditions

We focus on the hierarchy of factorization properties that **generalize** Unique Factorization—an area of research first introduced by Anderson, Anderson, and Zafrullah [1] in 1990.

Classical Finiteness Conditions

- **UF (Unique):** Exactly one factorization.
- **FF (Finite):** Atomic + finitely many factorizations.
- **BF (Bounded):** Atomic + factorization lengths are finite.
- **IDF (Irreducible Divisor):** Finitely many atomic divisors.
- **MCD-finite:** Finitely many maximal common divisors.

While the BF, IDF, and MCD-finite conditions are known to **generalize** the FF property, the precise distinctions between them have not been **rigorously and systematically established** until now.

Theorem. For each property $\mathscr{P}$ in {BF, IDF, MCD-finite}, there exists a monoid that satisfies $\mathscr{P}$ but neither of the other two properties.

Furthermore, the UF, FF, and BF properties **all contain the additional assumption of atomicity**. However, this assumption is somewhat arbitrary; indeed, because most domains are not atomic, **it is more natural to consider finiteness *separately* from atomicity**. This line of work was first investigated by Coykendall and Zafrullah [3], tackling a **generalization of the UF property**.

Definition (Coykendall & Zafrullah [3], 2004). A monoid M has the **unrestricted unique factorization** (U–UF) property if every **atomic** element has has exactly one factorization.

Motivated by the U–UF property, we introduce the U–FF condition, **a natural generalization of FF**. The U–FF property is more **fundamental** than classical conditions, as it **removes the assumption of atomicity**.

Beyond Atomicity

Central Definition. A monoid M has the **unrestricted finite factorization** (U–FF) property if every **atomic** element has finitely many factorizations.

The U–FF property **decouples finiteness from atomicity**.

Theorem. For a monoid M, the following statements are equivalent.
(a) M is a U–FF monoid.
(b) Every element of M has finitely many factorizations.
(c) The atomic submonoid of M is an FF monoid.

Theorem. If a monoid/domain is IDF, it must also be U–FF. However, if a monoid/domain is U–FF, it is not necessarily IDF.

In light of our results, the relationships between finiteness conditions can be summarized in the below **inclusion diagram**.

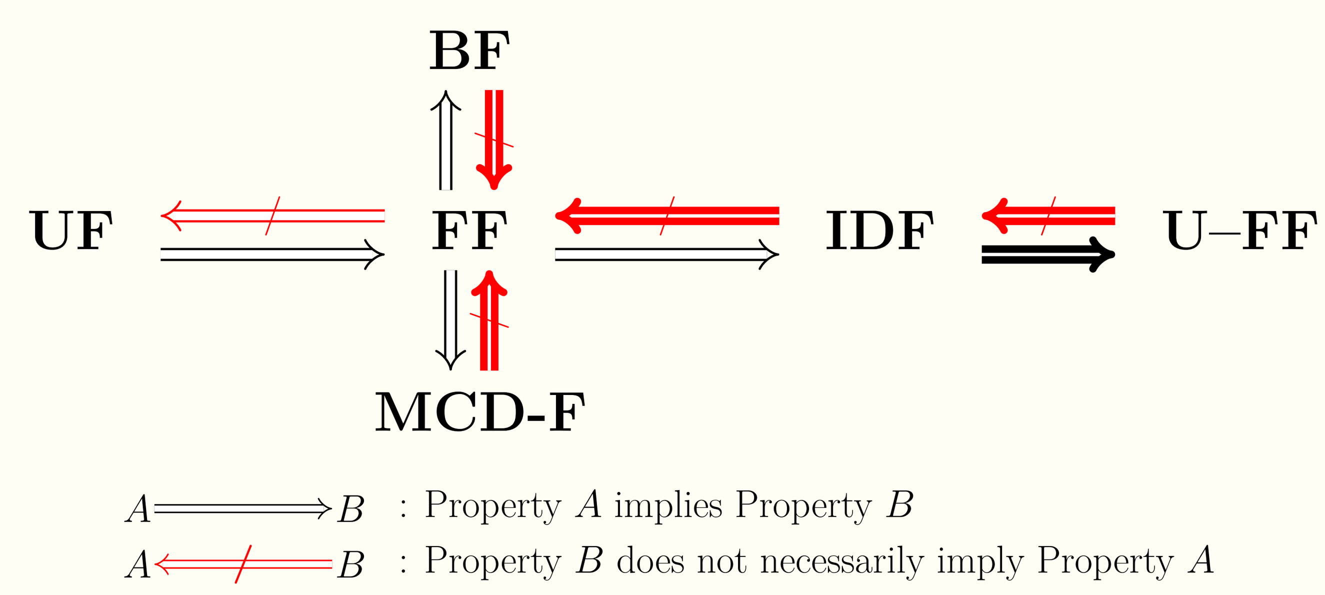

Diagram drawn by the student researcher using TikZ, 2026.

The U–FF property **strengthens a famous result** of Anderson et al. [1].

Theorem. For a monoid M, the following statements are equivalent.
(a) M is an FF monoid.
(b) M is an atomic IDF monoid.
(c) M is a nearly atomic U–FF monoid.

Polynomial Ascent

The ascent of IDF to polynomial extensions was first posed by Anderson et al. [1] in **1990**. This problem proved **challenging**—it was only recently solved by Eftekhari and Khorsandi [4] in **2018**, after **30 years**! As U–FF is a natural generalization of IDF, it is similarly important to consider the ascent of the U–FF property to polynomial extensions. **We tackle this generalized problem**, discovering **analogous results**.

Theorem. If R is a U–FF domain, then $R[x]$ is not necessarily a U–FF domain. If R is an MCD-finite U–FF domain, then $R[x]$ is always a U–FF domain.

References

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[3] J. Coykendall and M. Zafrullah, *AP-domains and unique factorization*, J. Pure Appl. **189** (2004) 27–35.

[4] S. Eftekhari and M. R. Khorsandi, *MCD-finite domains and ascent of IDF-property in polynomial extensions*, Comm. Algebra **46** (2018) 3865-3872.

[5] P. Malcolmson and F. Okoh, *Polynomial extensions of idf-domains and of idpf-domains*, Proc. Amer. Math. Soc. **137** (2009) 431–437.

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