# The Dual Canonical Basis in the Spin Representation via the Temperley-Lieb Algebra

## Background

**The Temperley-Lieb Algebra.** The Temperley-Lieb algebra, denoted by $\mathrm{TL}_n$, is the algebra generated by diagrams consisting of two parallel lines with $n$ vertices on the top and bottom lines, where vertices are connected by edges such that edges lie between the parallel lines, edges do not intersect, and each vertex is the endpoint of exactly one edge.

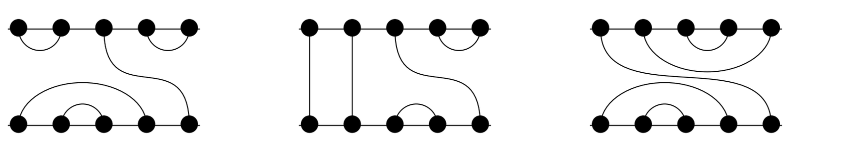

Figure 1. Examples of the diagrams that generate $\mathrm{TL}_5$. Created by the student researcher using TikZ, 2025.

Multiplication is defined by diagram concatenation with a factor of $\beta$, a formal variable, introduced for contractible loops.

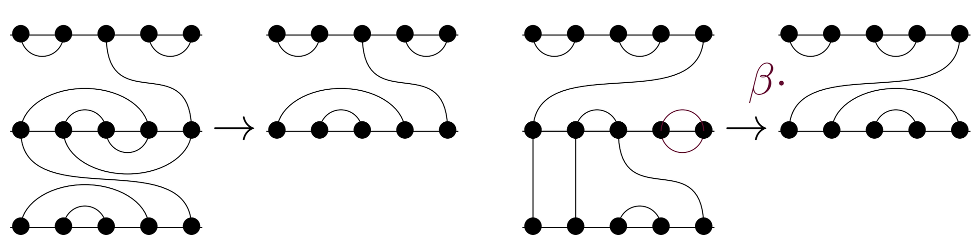


Figure 2. Examples of diagram concatenation in $\mathrm{TL}_5$. Created by the student researcher using TikZ, 2025.

**The Spin Representation.** The spin representation is $\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2 = (\mathbb{C}^2)^{\otimes n}$ and it parametrizes the spin states of $n$-electron systems. The spin representation is a representation of the Temperley-Lieb algebra, defined based on every "feature" that appears:

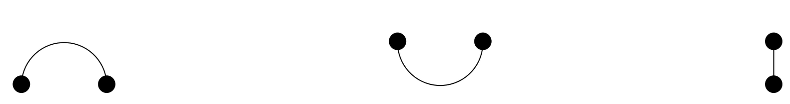

Figure 3. Possible types of features. Created by the student researcher using TikZ, 2025.

For each arc facing down connecting the $i$th and $j$th vertices, we apply $v_+ \otimes v_+ \mapsto 0, v_+ \otimes v_- \mapsto -q, v_- \otimes v_+ \mapsto 1, v_- \otimes v_- \mapsto 0$ to the $i$th and $j$th components. For each arc facing up connecting the $i$ and $i+1$th vertices, we apply $\delta\colon 1 \mapsto v_+ \otimes v_- - q^{-1} v_- \otimes v_+$ between the $i$ and $i+1$th components. For each line, we apply the identity.

The spin representation $(\mathbb{C}^2)^{\otimes n}$ has a natural basis $v_\pm \otimes \cdots \otimes v_\pm$, where $v_- = (0,1)$ and $v_+ = (1,0)$. Lusztig introduced a dual canonical basis for this representation. It has important applications in both mathematics and physics, including algebraic geometry, statistical mechanics, quantum field theory, and string theory. The elements of the dual canonical basis are labeled by $v_{k_n} \heartsuit \cdots \heartsuit v_{k_1}$ where $k_i \in \{-,+\}$ for all $i$. For $n = 2$, the dual canonical basis is

$$v_+ \otimes v_+, \quad v_- \otimes v_-, \quad v_- \otimes v_+, \quad v_+ \otimes v_- - q^{-1} v_- \otimes v_+.$$

For greater $n$, this basis gets more complicated; for example, $v_+ \heartsuit v_+ \heartsuit v_- \heartsuit v_- = v_+ \otimes v_+ \otimes v_- \otimes v_- - q^{-1}(v_+ \otimes v_- \otimes v_+ \otimes v_-) - q^{-1}(v_- \otimes v_+ \otimes v_- \otimes v_+) + q^{-2}(v_- \otimes v_- \otimes v_+ \otimes v_+)$. Here, $q \in \mathbb{C}$ is a parameter, the quantum deformation, which gives a family of dual canonical bases.

## The Physical Overview

Quantum mechanics is the study of Hilbert spaces and how Hamiltonians act on them. When a Hamiltonian acts on a Hilbert space, its eigenvectors represent stationary quantum states that are invariant over time, and its eigenvalues describe the possible energy levels of the system. The spin representation $(\mathbb{C}^2)^{\otimes n}$ is an important finite-dimensional Hilbert space, with an action of the quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$, which appears in quantum integrable systems and provide solutions to the Yang-Baxter equation. The Cartan element, $h \in \mathcal{U}_q(\mathfrak{sl}_2)$, is an important Hamiltonian in the quantum group. However, it does not act with simple spectrum: it divides the spin representation into eigenspaces, but just by considering the action of $h$, it is impossible to further distinguish within each eigenspace. Our research resolves this problem through introducing another mathematical object, the Temperley-Lieb algebra, whose action commutes with that of $\mathcal{U}_q(\mathfrak{sl}_2)$. We use diagrams of the Temperley-Lieb algebra to construct a visual basis of the spin representation that respects the action of $h$. These diagrams allow us to distinguish the structure of each eigenspace, providing a complete basis of the spin representation that is visual and simple to work with. Furthermore, this diagrammatic realization also leads to an explicit formula for the basis vectors, enabling concrete computations.


**References.** Galyna Dobrovolska, Vinoth Nandakumar, and David Yang. "Modular representations in type A with a two-row nilpotent central character." In: *Journal of Algebra* **643** (2024), pp. 311–339. C. K. Fan and Richard M. Green. "Monomials and Temperley–Lieb algebras." In: *Journal of Algebra* **190.2** (1997), pp. 498–517. Igor B. Frenkel, Mikhail G. Khovanov, and Alexandre A. Kirillov Jr. "Kazhdan–Lusztig polynomials and canonical basis." In: *Transformation Groups* **3.4** (1998), pp. 321–336. Mikhail Khovanov. *Graphical calculus, canonical bases and Kazhdan–Lusztig theory*. Yale University, 1997.


## Main Result

The following $\mathrm{TL}_n$-modules are isomorphic:

$$\bigoplus_{0\le k\le n} \mathrm{Ind}_{\mathrm{TL}_k\otimes\mathrm{TL}_{n-k}}^{\mathrm{TL}_n} \mathbb{C}_{\mathrm{triv}} \cong (\mathbb{C}^2)^{\otimes n}$$

and the isomorphism identifies the diagrammatic basis of the left hand side to the dual canonical basis of the right hand side through the diagrammatic action on the spin representation. This result is constructive: given any dual canonical basis element, we can construct a diagram, and vice versa.

**Proof Outline of Main Result.** The proof of the main result heavily relies on an intermediate object called the Hecke algebra, denoted $\mathcal{H}$. We first prove that $\mathrm{Ind}_{\mathcal{H}_k\otimes\mathcal{H}_{n-k}}^{\mathcal{H}_n} \mathbb{C}_{\mathrm{triv}} \cong \mathrm{Ind}_{\mathrm{TL}_k\otimes\mathrm{TL}_{n-k}}^{\mathrm{TL}_n} \mathbb{C}_{\mathrm{triv}}$, then demonstrate that all arrows in the following commutative diagram map a canonical basis element to either another canonical basis element or $0$, where $\mathrm{Ind}_{\mathcal{H}_k\otimes\mathcal{H}_{n-k}}^{\mathcal{H}_n} \mathbb{C}_{\mathrm{triv}}$, also denoted $\mathcal{M}$, is the spherical module.

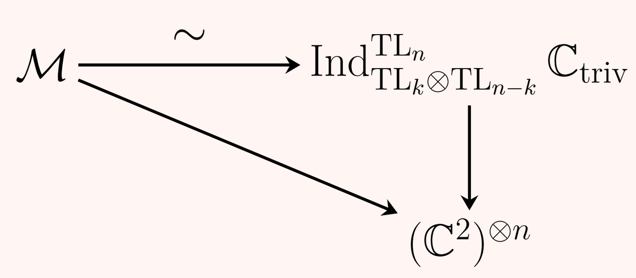


Figure 4. A commutative diagram describing the proof outline of main result. Created by the student researcher using TikZ, 2025.

## Explicit Construction of Main Result

Let $(\mathbb{C}^2)_k^{\otimes n}$ denote the subspace of the spin representation generated by the basis elements with $k$ instances of $v_-$ and $n-k$ instances of $v_+$. Given a label of the dual canonical basis $v_{k_n} \heartsuit \cdots \heartsuit v_{k_1}$ in $(\mathbb{C}^2)_k^{\otimes n}$, we construct an unique diagram $D$ such that the main result identifies $D$ with $v_{k_n} \heartsuit \cdots \heartsuit v_{k_1}$.

**From Label to Diagram.** First, we draw the quasi-simple links on the top line. Label the vertices from left to right by $v_{k_n}, v_{k_{n-1}}, \ldots, v_{k_1}$. We construct these quasi-simple links by the following procedure.

1. Consider the least $i$ such that $k_i = +$ that we have not considered yet.
   - If there is no $j < i$ such that $k_j = -$, move on to step (2).
   - If $j$ is the greatest $j < i$ such that $k_j = -$, draw a quasi-simple link between $v_{k_i}$ and $v_{k_j}$ and move on to step (2).
2. Repeat step (1) until we have considered all $v_+$.

After this step, let the number of quasi-simple links on the top line be $p$. On the bottom line, we draw the quasi-simple link connecting the $(k-i)$th vertex to the $(k+1+i)$th for all $0 \le i \le p-1$, and so on, drawing $p$ quasi-simple links total. Finally, we have $n-2p$ unlinked vertices on both the top and the bottom lines. We link the $i$th unlinked vertex from the left on the top line to the $i$th unlinked vertex from the left on the bottom line for $1 \le i \le n-2p$, obtaining our diagram.

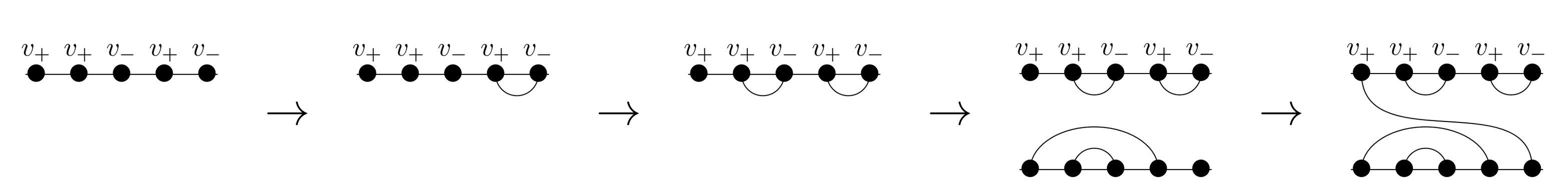


Figure 5. Construction of the diagram corresponding to $v_+ \heartsuit v_+ \heartsuit v_- \heartsuit v_+ \heartsuit v_-$. Created by the student researcher using TikZ, 2025.

This process is easily invertible, and there is a similar constructive process to determine a label of a dual canonical basis given a diagram.

## Explicit Dual Canonical Basis Formula

With our main result and its construction, we devise the first explicit, non-inductive formula for the dual canonical basis. We prove that for $D \in \mathrm{Ind}_{\mathrm{TL}_k\otimes\mathrm{TL}_{n-k}}^{\mathrm{TL}_n} \mathbb{C}_{\mathrm{triv}}$, the dual canonical basis corresponding to $D$ is equal to

$$\sum_{I\in T(D)} \mathrm{sgn}(I) v_I$$

where $T(D)$, $\mathrm{sgn}$, and $v_I$ can be explicitly computed from $D$. This is the first non-inductive formula for the dual canonical basis.

## Demonstrating Compatibility, Other Results, and Future Work

We use our diagrammatic perspective to reexamine known results and develop new results.

**The isomorphism $\mathrm{End}_{\mathbf{U}}((\mathbb{C}^2)^{\otimes n}) \cong ((\mathbb{C}^2)^{\otimes 2n})^{\mathbf{U}}$ through $\mathrm{TL}_n$.** This known isomorphism was originally proven by Khovanov (1997) through a complicated algebraic argument. We reprove it through a visual argument involving the composition of diagrams.

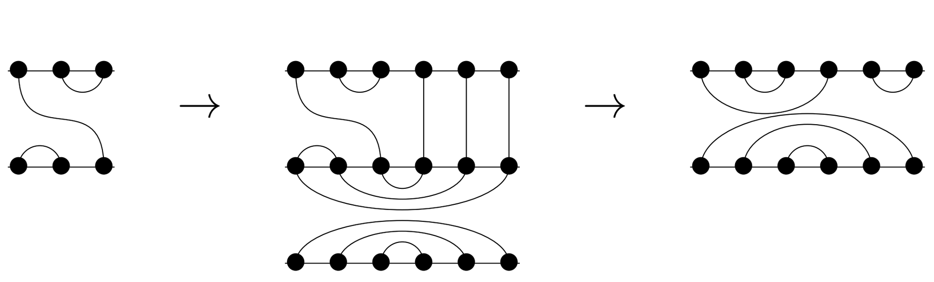

Figure 6. An example of the isomorphism for $n = 3$, expressed diagrammatically. Created by the student researcher using TikZ, 2025.

**The duality of the canonical and dual canonical bases.** We verify that the canonical and the dual canonical bases of the spin representation are indeed dual to each other through the perspective of the Hecke algebra. Through doing so, we obtain new results on the dual spaces to the spherical and aspherical modules ($\mathcal{M}^*$ and $\mathcal{N}^*$, respectively) as a byproduct. Some of the results are:

- that $\underline{Q}_w$ is equal to $\mathrm{flip}(\underline{M}_{w_f w})$ under the identification $M_w \mapsto Q_w$.
- that $\underline{R}_w$ is equal to $\mathrm{flip}(\underline{N}_{w_f w})$ under the identification $N_w \mapsto R_w$.

**Alternative axiomatic description of the canonical basis.** We also propose an alternative axiomatic description of the canonical basis of the spin representation which allows us to view the canonical basis diagrammatically.

**Explicit Irreducible Decomposition of the Spin Representation of the Temperley-Lieb Algebra.** Another one of my papers uses our diagrammatic perspective of the spin representation to develop an explicit formula for the decomposition of the spin representation into Specht modules, explicitly describing $(\mathbb{C}^2)_k^{\otimes n} \cong W_n^n \oplus \cdots \oplus W_{n-2k}^n$ by viewing the left-hand side as Temperley-Lieb diagrams, allowing us to view multiple-electron systems explicitly as multiple single-particle systems.

**Future Work in the Affine Setting.** Currently, we are studying the analog of the spin representation in the affine setting, in which $(\mathbb{C}^2)^{\otimes n}$ is replaced by $((\mathbb{C}^2)^{\otimes n})[t^{\pm 1}]$ and $\mathrm{TL}_n$ is replaced by the affine Temperley-Lieb algebra, generated by diagrams on a cylinder. In the affine spin representation, there is no axiomatic definition of the dual canonical basis.


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