Tetrahedron-intersecting families of 3-uniform hypergraphs

Background

Definition. A graph is a collection of vertices and edges. Each edge links a pair of vertices together.

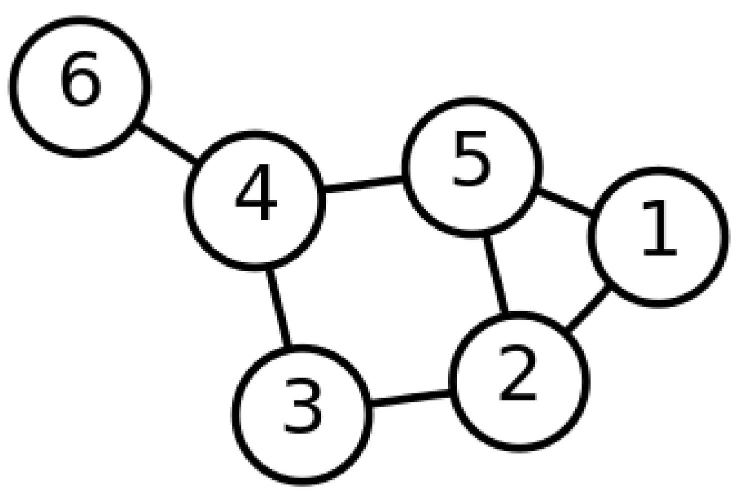

Figure 1. A graph with 6 vertices and 7 edges. Image from Wikipedia, 2007. `en.wikipedia.org/wiki/Graph_(discrete_mathematics)`

Graphs are used to represent objects and their pairwise relationships. Often, we wish to analyze the similarities between two graphs.

Definition. The intersection of two graphs is the graph formed by the edges common to both graphs.

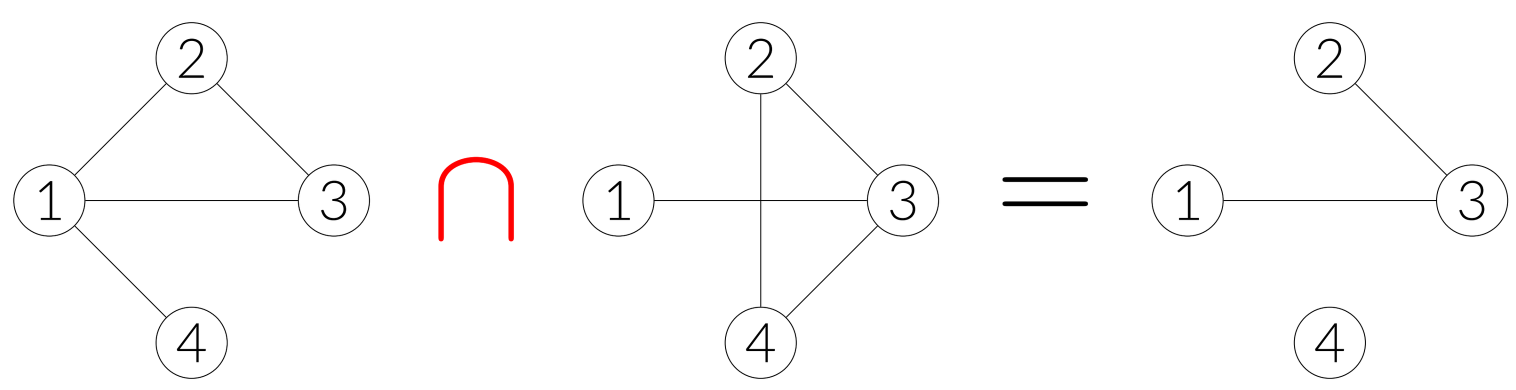

Figure 2. The intersection of two labeled graphs on 4 vertices. Image by Finalist using TikZ, 2025.

In some cases, we wish to represent relationships between multiple vertices. This leads to a natural generalization of a graph, called a hypergraph, where edges can join more than two vertices. Formally,

Definition. A hypergraph $H = (V, E)$ is a collection of vertices V and edges E, where each edge is a subset of V. If every edge contains k vertices, then we say H is k-uniform.

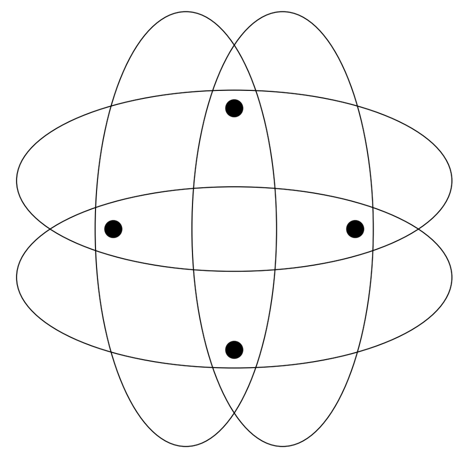

Figure 3. An unlabeled 3-uniform hypergraph with 4 vertices and 4 edges. This particular 3-uniform hypergraph is called a tetrahedron. Image by Finalist using TikZ, 2024.

Previous Work

The study of intersection problems is an active and important area in extremal combinatorics and theoretical computer science.

Definition. A family of graphs is triangle-intersecting if the intersection of any two graphs contains some triangle as a subgraph.

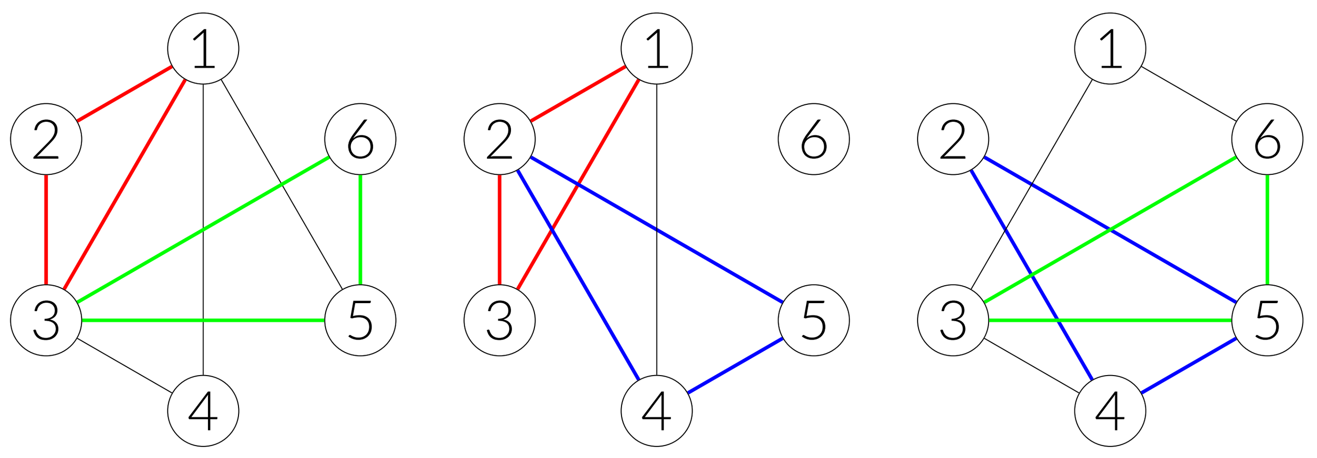

Figure 4. A triangle-intersecting family of graphs on 6 labeled vertices. Image by Finalist using TikZ, 2024.

Question. What is the maximum size of a triangle-intersecting family of graphs on n labeled vertices?

Surprisingly, this seemingly straightforward problem proved to be notoriously challenging! In **1976**, Simonovits and Sós conjectured an answer of $2^{\binom{n}{2}-3}$. However, it wasn't until **2012**—**36** years later—when Ellis, Filmus, and Friedgut verified the Simonovits-Sós conjecture.

Since then, other results of a similar flavor have emerged. In 2021, Berger and Zhao proved the maximal size of a K_4-intersecting family of graphs.

Hypergraph Families

In my work, I solved a question analogous to the the Simonovits-Sós conjecture in the hypergraph setting, proving the maximal size of a tetrahedron-intersecting family of 3-uniform hypergraphs. To the best of my knowledge, this is the first resolution of this type of intersection problem in the hypergraph setting.

Main Theorem

Let $\mathcal{F}$ be a tetrahedron-intersecting family of 3-uniform hypergraphs on n labeled vertices.

- **(Upper bound)** $|\mathcal{F}| \leq 2^{\binom{n}{3}-4}$.
- **(Uniqueness)** The upper bound is an equality if and only if $\mathcal{F}$ consists of all 3-uniform hypergraphs containing some fixed tetrahedron.

Solution Outline

It's easy to see the upper bound is tight; just take all the 3-uniform hypergraphs containing some fixed tetrahedron.

Previous results introduced a framework that transforms this type of intersection problem into an infinite linear program. More precisely, the problem is reduced into proving that there exists a set of coefficients $\{c_H\}$, indexed by non-isomorphic unlabeled 3-uniform hypergraphs, with $c_\varnothing = 2^4 - 1$, such that for a certain probabilistic graph transformation function f, the sum

$$\mu(G) := \sum_H c_H \cdot \mathbb{P}[f(G) \cong H]$$

satisfies $|\mu(G)| \leq 1$ for all non-empty G.

I used CVXPY, a convex programming solver, to generate a conjectured solution for $\{c_H\}$.

To verify the conjectured solution to the infinite linear program, f has the key property where that as G grows large, $f(G)$ also tends to be large. Hence, for fixed H the probability $\mathbb{P}[f(G) \cong H]$ decays, and so $\mu(G)$ also decays as G grows large.

The idea is therefore to prove some bound that verifies $|\mu(G)| \leq 1$ for all sufficiently large G, and verify the remaining hypergraphs via a direct computation of μ.

However, this is especially difficult for 3-uniform hypergraphs given the super-exponential rate at which their numbers grow (e.g. there are over 10^{12} non-isomorphic 3-uniform hypergraphs on 8 vertices, compared to just 12346 graphs). So, much tighter bounds were required compared to previous results to keep the final computation feasible.

The bulk of my work therefore consists of a careful analysis of f and hypergraph intersections to derive sufficiently tight bounds. In the end, I showed that $|\mu(G)| < 1$ if any subgraph of G satisfied some certain properties. I then devised, proved, and implemented an algorithm that identifies a collection of graphs satisfying these properties. Moreover, the algorithm guarantees that any sufficiently large G must contain at least one of these graphs as a subgraph, hence verifying $|\mu(G)| < 1$. Finally, I directly computed $\mu(G)$ for all remaining G in C++, completing the proof.

Further Directions

Question. What is the maximum size of a $K_4^{(3)-}$-intersecting family of 3-uniform hypergraphs on n labeled vertices?

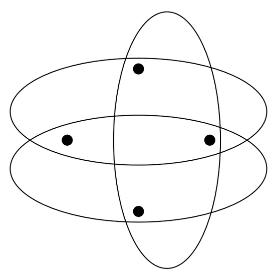

Figure 5. An unlabeled $K_4^{(3)-}$ (a tetrahedron with one edge removed). Image by Finalist using TikZ, 2024.

Question. What is the maximum size of a K_t-intersecting family of graphs on n labeled vertices, in terms of n and t?