Concatenations of Incidence Matrices, Spanning Trees in Planar Graphs: Two Related Problems

Introduction

Problem 1. What is the largest number of spanning trees τ_m you can have in a planar graph with m edges?

Problem 2. Given a square bi-incidence matrix, what is its largest possible determinant $\max\det_m$?

m	1	2	3	4	5	6	7	8	9	10
τ_m	1	2	3	5	8	16	24	45	75	130
$\max\det_m$	1	2	3	5	8	16	24	45	75	130

- The problem of counting spanning trees is central to the field of counting and sampling, and has applications to telecommunications networks and geometry.
- The problem of finding the number of spanning trees in a planar graph was first introduced by statistical physicists to model Ferromagnetism.
- Ice-type models, the Potts model, and the Ising model are all examples of physical ensemble models closely related to the number of spanning trees in a planar graph.

Definitions

Graphs are abstract mathematical objects used to describe sets of objects (as vertices) along with relationships between them (as edges). Some examples include:

- A railway system between cities
- An electrical circuit network
- Irrigation canals and waterways

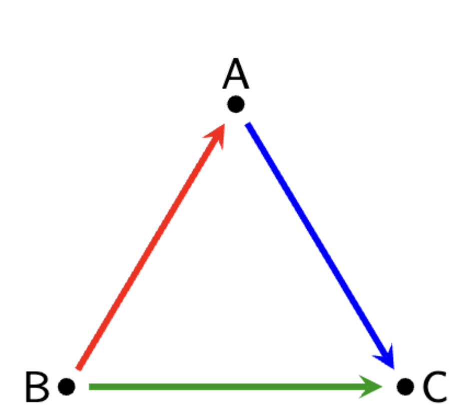
Definition. A **graph** $G = (V, E)$ consists of a set of vertices V along with a set of edges E between those vertices. We say that G is **planar** if it can be drawn in a plane so that any two edges can only intersect at a vertex of G.

Given a planar directed graph G, we can construct its **directed planar dual** G^* by placing the vertices of G^* in the faces of some planar embedding of G (including the outer face). For each edge e in G, we obtain a new edge in G^* by "rotating" the edge e by 90° counterclockwise.

Definition. A **spanning tree** on a graph G is a minimal set of edges that connects all of the vertices of G.

We denote the total number of different spanning trees of a graph G by $\tau(G)$ and define τ_m to be the largest possible number of spanning trees in a planar graph with m edges.

Definition. Given a directed graph, its **incidence matrix** is a matrix where each edge is represented by a row with ± 1 in the columns corresponding to endpoints of the edge.



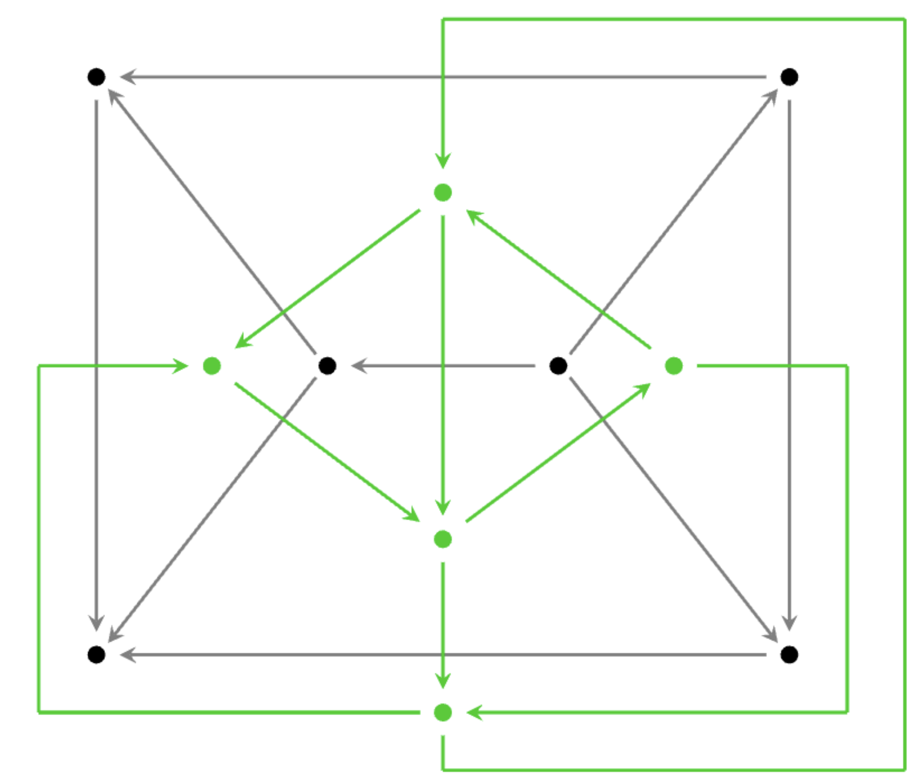
A	B	C
1	-1	0
-1	0	1
0	-1	1

We say a matrix is an **incidence submatrix** if it can be obtained by removing some rows and columns from an incidence matrix. We call a square matrix that can be obtained by concatenating, or putting side-by-side, two incidence submatrices a **bi-incidence matrix**, and define $\max\det_m$ to be the largest determinant of a $m \times m$ bi-incidence matrix.

A Linear Algebraic Connection

The key to proving $\tau_m \leq \max\det_m$ lies in the following construction using planar duality.

Theorem 1.



$$\det\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & -1 \\ 1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 \end{bmatrix} = \tau(\mathbf{G}) = \tau(\mathbf{G^*})$$

Theorem 2. For sufficiently large $m \in \mathbb{N}$,

$$1.791^m \leq \tau_m \leq \max\det_m \leq \left(\sqrt[3]{7}\right)^m \simeq 1.913^m.$$

In particular, the third and fourth inequalities hold for all $m \in \mathbb{N}$.

Conjecture 1. $\tau_m = \max\det_m$ for all positive integers m.

A New Planarity Criterion

Definition. We define the **excess** of a graph G to be equal to $\varepsilon(G) = \tau(G) - \max\det(G)$, where $\max\det(G)$ denotes the largest determinant of the concatenation of the incidence submatrix of G with a incidence submatrix.

Theorem 3. Let G be a graph. Then

$$\varepsilon(G) \begin{cases} = 0, & \text{if } G \text{ is planar,} \\ \geq 18, & \text{otherwise.} \end{cases}$$

In other words, G is planar if and only if $\varepsilon(G) = 0$.

Future Directions

Theorem 4. If G is a subdivision of $K_{3,3}$ or K_5 with m edges, then $\max\det G \leq \tau_m$.

We prove Theorem 4 using a method called the edge-relocation method, eliminating an infinite class of nonplanar graphs from being counterexamples to the conjecture.

Some potential approaches to proving the conjecture include closing the gap between the lower and upper bounds in Theorem 2 or generalizing Theorem 4 to include all nonplanar graphs.

We hope that our work and methods inspire future research to resolve this conjecture and gain deeper insight into this surprising connection across fields.

*All images used are produced by the author.