Factorizations in Evaluation Monoids of Laurent Semirings

Background

- In 300 BC, Euclid proved that the integers, $\mathbb{Z}$, have unique prime factorization. In larger sets like $\mathbb{Z}[\sqrt[3]{3}]$ or $\mathbb{Z}[i]$, however, factorization may no longer be unique.
- This study of characterizing when non-unique factorizations in *rings* occur is key to the field of algebraic number theory. My project furthers this study for Laurent evaluation monoids.

Notation

An (additive) **monoid** is a pair $(M, +)$, where M is a set and $+$ is a binary operation on M, such that

- $+$ is both associative and commutative, and
- there exists $0 \in M$ such that $x + 0 = x$.

$\mathbb{Z}$

An integer $p \geq 2$ is a *prime* if $p = a \cdot b$ for any $a, b \in \mathbb{Z}_{\geq 1}$ implies $a = 1$ or $b = 1$.

Monoid

An element $a \neq 0$ in $(M, +)$ is an **atom** if $a = x + y$ for some $x, y \in M$ implies $x = 0$ or $y = 0$.

$\mathbb{Z}$

Fundamental Theorem of Arithmetic: Every $n \in \mathbb{Z}_{\geq 2}$ factors (uniquely) into primes.

Monoid

$(M, +)$ is **atomic** if every nonzero element can be written as a sum of atoms.

- Denote by $\mathcal{A}(M)$ the set of atoms in M.
- A **factorization** of a nonzero $x \in M$ is a decomposition $x = a_1 + \cdots + a_\ell$, where $a_1, \dots, a_\ell \in \mathcal{A}(M)$, in which case ℓ is called a **length** of x.
- Denote by $\mathsf{Z}(x)$ the set of factorizations of x and by $\mathsf{L}(x)$ the set of lengths of x.

Based on "UFD" notion.

- M is a **bounded factorization monoid** (BFM) if for each nonzero $x \in M$, the set $\mathsf{L}(x)$ is bounded.
- M is a **finite factorization monoid** (FFM) if each nonzero $x \in M$ has finitely many factorizations.
- M is a **unique factorization monoid** (UFM) if each nonzero $x \in M$ has exactly one factorization.

Based on "Noetherian ring" notion. M satisfies the **ascending chain condition on principal ideals** (ACCP) if every sequence $\{x_n\} \subseteq M$ satisfying $x_n - x_{n+1} \in M$ for each $n > 0$ is constant after some point.

The following diagram of implications is a well-established diagram that holds true for any monoid.

$$\begin{array}{c}\textbf{UFM}\\ \Downarrow \\ [\textbf{FFM}, \textbf{HFM} \Rightarrow \textbf{BFM} \Rightarrow \textbf{ACCP}] \\ \Downarrow \\ \textbf{atomicity}\end{array}$$

Objective

What can we say about the existence and non-uniqueness of factorizations in the Laurent evaluation monoid $M_\alpha := \mathbb{Z}_{\geq 0}[\alpha, \alpha^{-1}]$, where $\alpha \in \mathbb{R}_{>0}$?

Results

We begin by characterizing when M_α is atomic; that is, when factorization "exists" in M_α.

Theorem 1. The following statements are equivalent.

(a) $1 \in \mathcal{A}(M_\alpha)$.
(b) $\mathcal{A}(M_\alpha) = \{\alpha^n \mid n \in \mathbb{Z}\}$.
(c) M_α is atomic.

We then show that the reverse implications in the second line of the diagram are also true.

Theorem 2. The following statements are equivalent.

(a) M is a BFM.
(b) M is an FFM.
(c) M satisfies the ACCP.

Lastly, we find a class of monoids that are FFMs but not UFMs using the theorem below. This shows that the implication from the first to the second line of the diagram is not reversible.

Theorem 3. Suppose that α_1 and α_2 are the roots of an irreducible quadratic polynomial in $\mathbb{Q}[x]$ such that $0 < \alpha_1 < 1 < \alpha_2$. Then M_{α_1} is an FFM and, therefore, satisfies the ACCP.

Conclusions

We answer the question **What can we say about factorizations in monoids?** for Laurent evaluation monoids, refining the diagram on the left.

$$\begin{array}{c}[\textbf{UFM} \Leftrightarrow \textbf{HFM} \Leftrightarrow \textbf{LFM}] \\ \Downarrow \\ [\textbf{FFM} \Leftrightarrow \textbf{BFM} \Leftrightarrow \textbf{ACCP}] \\ \Downarrow \\ \textbf{atomicity}\end{array}$$

HFMs (half-factorial monoid) and LFMs (length-factorial monoid) are other types of atomic monoids characterized by how "unique" the lengths of their factorizations are.

Future work that could be done include

- Refining complete diagram for $\mathbb{N}_0[\{a_i \mid i \in I\}]$, where a_i are complex numbers, both multiplicatively and additively
- Determine when M_α is atomic using only α's minimal polynomial

All graphics are created by the researcher.