On the Smoothness and Regularity of the Chess Billiard Flow and the Poincaré Problem

Introduction

- Internal waves are important to the study of oceanography and to the theory of rotating fluids.
- They describe how an originally unmoving fluid can move and evolve under perturbation by a periodic forcing function.
- We study the behavior of a particular two-dimensional model for these internal waves, called the Poincaré problem, which forms patterns called billiard flows.
- Billiard flows map points on a boundary of a given shape to another point on that boundary, with some sort of reflection or bouncing step (e.g. in a game of pool).

The chess billiard map preserves the slopes of the trajectories. Each mapping $b$ consists of traveling first on a line of slope $\rho$ (blue), and then bouncing off the boundary at a slope of $-\rho$ (red).

The chess billiard map can be more formally described as traveling on lines of slope $\rho = \frac{1 - \lambda^2}{\lambda}$ and $-\rho = -\frac{1 - \lambda^2}{\lambda}$, for $0 < \lambda < 1$. This represents one iteration of the mapping.

Rotation in the Square

The definition of rotation number $r$ can be thought of as point $x$ being rotated counterclockwise to $b(x)$.

We quantify the rotation using the rotation number $r$, or the average rotation per mapping over time. For example, in the above figure, the rotation number is $r = \frac{1}{4}$.

"Rational" and "irrational" rotation correspond directly to the rationality or irrationality of $r$. When $r$ is rational, a period trajectory forms — the same lines are traveled along again and again. However, in an irrational rotation (i.e. $r$ is irrational), a point can never be mapped to itself again, so there is no periodic trajectory.

The wave equation

I studied the wave equation known as the Poincaré problem, which models the chess billiard maps:

$$\left(\partial_t^2 + \partial_{x_1}^2 + \partial_{x_2}^2\right)u = f(x) \cos \lambda t,$$

$$u|_{t=0} = \phi, \quad u_t|_{t=0} = 0, \quad u|_{\partial \Omega} = 0,$$

where $\Omega$ is a smooth 2D convex domain with boundary $\partial \Omega$, $f(x)$ is the forcing function, and $\lambda \in (0, 1)$ is the frequency of the periodic forcing in $\cos (\lambda t)$.

- $u$ represents the chess billiard map and is the solution to the Poincaré problem.
- $u$ is the 'stream function' and maps the fluid’s velocity in two dimensions.
- It is what I show is highly differentiable (smooth).

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Statement of Main Result

Our main result is: Given a Diophantine-irrational rotation of the chess billiard map in the square, the solution to the wave equation is smooth, or infinitely differentiable.

More formally, we have the following theorem, where $C^\beta$ represents $\beta$-times differentiable.

Theorem. Given a forcing function $f(x) \in C^\beta[0,1] \times [0,1]$ and a $\beta$-Diophantine rotation number $r(\lambda)$ for some $\beta$ and $C > 0$, the solution $u(t)$ of the Poincaré problem in the square is in $C^{\beta - 1}$. 

Proof. A rough roadmap of the proof is

- Write $u$ as the sum of exponentials (specifically, as its Fourier decomposition)
- Bound these sums (using the Diophantine irrational condition)
- Relate the bounds to smoothness (specifically, using Sobolev spaces; bounded Fourier coefficients can translate to higher differentiability)

Corollary. If $f$ is smooth (i.e. $f \in C^\infty$), then $u$ is also smooth.

Future Directions

The natural extension of this results is to examine the behavior of $u$ and irrational rotation in shapes beyond the square.

Conjecture. For all convex shapes, $u$ is smooth for sufficiently irrational rotation.

- Studying the wave behavior is more difficult in shapes where rotation numbers cannot be explicitly calculated.
- I ran numerical simulations to estimate $r(\lambda)$ at different $\lambda$ for ’perturbed’ squares, such as a trapezoid, tilted square, and rounded square.

I’d like to perform more simulations to determine approximate $r(\lambda)$ and analyze rapid decay of the Fourier coefficients beyond the square.