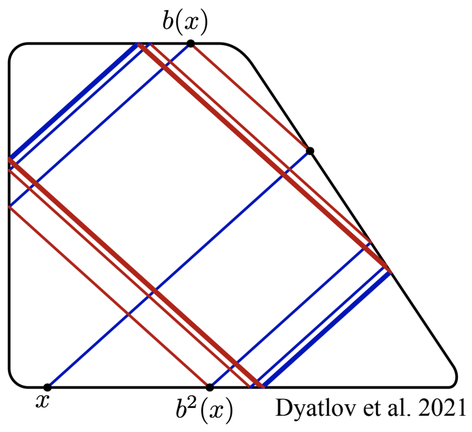
On the Smoothness and Regularity of the Chess Billiard Flow and the Poincaré Problem

Introduction

- Internal waves are important to the study of oceanography and to the theory of rotating fluids.
- They describe how an originally unmoving fluid can move and evolve under perturbation by a periodic forcing function
- We study the behavior of a particular two-dimensional model for these internal waves, called the Poincaré problem, which forms patterns called billiard flows
- Billiard flows map points on a boundary of a given shape to another point on that boundary, with some sort of reflection or bouncing step (ex: in a game of pool)

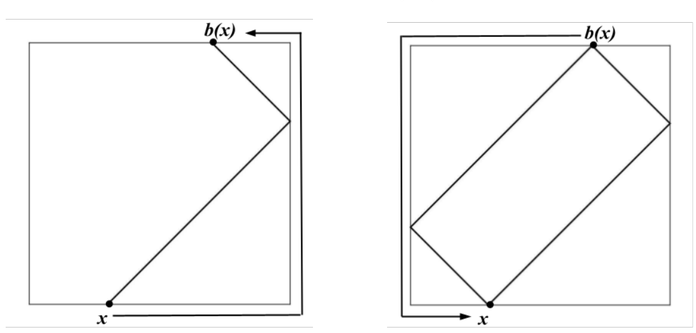
The *chess billiard map* preserves the slopes of the trajectories. Each mapping b consists of traveling first on a line of slope ρ (blue), and then bouncing off the boundary at a slope of $-\rho$ (red).



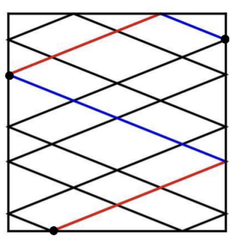
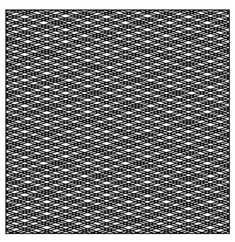
Dyatlov et al. 2021

The Chess Billiard Map and Rotation

We can describe the chess billiard map as a rotation of points along the boundary; the mapping from x to $b(x)$ below can be thought of as point x being rotated counterclockwise to $b(x)$.



We quantify the rotation using the *rotation number* r, or the average rotation per mapping over time. For example, in the above figure, the rotation number is $r = \frac{1}{2}$.
"Rational" and "irrational" rotation correspond directly to the rationality or irrationality of r. When r is rational, a period trajectory forms — the same lines are traveled along again and again. However, in an irrational rotation (i.e. r is irrational), a point can never be mapped to itself again, so there is no periodic trajectory.

Here, the rational rotation (left) has clear lines, while the near-irrational (right) is almost fully shaded and smooth. In my project, I show that in the case of irrational rotation, the flow becomes mathematically smooth.

The Wave Equation

I studied the wave equation known as the Poincaré problem, which models the chess billiard maps:

$$(\partial_t^2(\partial_{x_1}^2 + \partial_{x_2}^2) + \partial_{x_2}^2)u = f(x)\cos\lambda t,$$
$$\mathrm{u}|_{t=0} = \partial_t u|_{t=0} = 0, \qquad u|_{\partial\Omega} = 0,$$

where Ω is a smooth 2D convex domain with boundary $\partial\Omega$, $f(x)$ is the forcing function, and $\lambda \in (0, 1)$ is the frequency of the periodic forcing in $\cos(\lambda t)$.

- u represents the chess billiard map and is the solution to the Poincaré problem.
- u is the "stream function" and maps the fluid's velocity in two dimensions.
- It is what I show is highly differentiable (smooth).

All uncited images and figures created by author.

Rotation in the Square

The chess billiard mapping can be more formally described as traveling on lines of slope $\rho = \frac{\sqrt{1-\lambda^2}}{\lambda}$ and $-\rho = -\frac{\sqrt{1-\lambda^2}}{\lambda}$, for a $0 < \lambda < 1$. This represents one iteration of the mapping.

Definition. The *rotation number* of a chess billiard map with parameter λ and starting point p over n iterations is

$$r(p, \lambda) = \lim_{n\to\infty} \frac{d_n}{n}$$

where d_n is total fractional distance traveled around the boundary.

Proposition. The rotation number $r(\lambda)$ of the chess billiard map in a square is $\frac{\lambda}{\sqrt{1-\lambda^2}+\lambda}$.

We use the following irrational rotation condition on $r(\lambda)$.

Definition. Let r be an irrational number. We call r *$\beta-$Diophantine* if for all rationals p/q, with $q \in \mathbb{Z}^+$, there exist some constants β and $C > 0$, for which r satisfies the inequality

$$\left| r - \frac{p}{q} \right| \geq \frac{C}{q^{2+\beta}}.$$

- Loosely, r being Diophantine means that it is far from all rationals.

Statement of Main Result

Our main result is: Given a Diophantine-irrational rotation of the chess billiard map in the square, the solution to the wave equation is smooth, or infinitely differentiable.
More formally, we have the following theorem, where C^q represents q-times differentiable.

Theorem. Given a forcing function $f(x) \in C^s[0,1] \times [0,1]$ and a β-Diophantine rotation number $r(\lambda)$ for some β and $C > 0$, the solution $u(t)$ of the Poincaré problem in the square is in $C^{s-1-\beta}$.

Proof. A rough roadmap of the proof is

- Write u as the sum of exponentials (specifically, as its Fourier decomposition)
- Bound these sums (using the Diophantine irrational condition)
- Relate the bounds to smoothness (specifically, using Sobolev spaces; bounded Fourier coefficients can translate to higher differentiability)

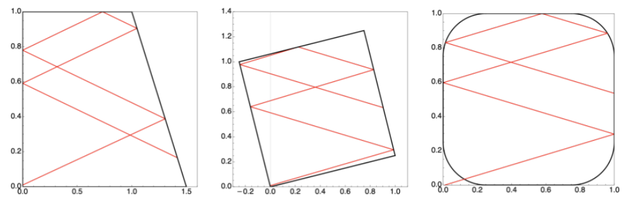
Corrollary. If f is smooth (i.e. $f \in C^\infty$), then u is also smooth.

Future Directions

The natural extension of these results is to examine the behavior of u and irrational rotation in shapes beyond the square.

Conjecture. For all convex shapes, u is smooth for sufficiently irrational rotation.

- Studying the wave behavior is more difficult in shapes where rotation numbers cannot be explicitly calculated.
- I ran numerical simulations to estimate $r(\lambda)$ at different λ for "perturbed" squares, such as a trapezoid, tilted square, and rounded square.



I'd like to perform more simulations to determine approximate $r(\lambda)$ and analyze rapid decay of the Fourier coefficients beyond the square.