A Generalization of q-Calculus Using Formal Group Laws

Abstract

In a field of math called q-calculus, there is an operator called the q-derivative, which is analogous to the derivative from calculus. We generalized the q-derivative to an operator called the s-derivative by modifying a formula called the q-power rule. Several theorems in calculus and q-calculus generalize to s-calculus. Also, we defined the s-binomial coefficients (which generalize the q-binomial coefficients) and proved analogues of combinatorial identities for the s-binomial coefficients.

Background: What is q-calculus?

One of the most important ideas in calculus is the derivative, which measures the rate of change of a function. In another area of math called q-calculus, there is an operator called the q-derivative, which is similar to the derivative in many ways. The q-derivative of a function f at a point x is defined by

$$\frac{f(qx) - f(x)}{qx - x},$$

and as q gets closer and closer to 1, the q-derivative approaches the usual derivative of f.

Example

The q-derivative of x^n is

$$\frac{q^n x^n - x^n}{qx - x} = \left(\frac{q^n - 1}{q - 1}\right) x^{n-1}.$$

This fact is called the q-power rule.

We use the symbol $[n]_q$ to denote $(q^n - 1)/(q - 1)$, and we call it the ***q*-analogue of *n***. This is because the q-derivative of x^n is $[n]_q x^{n-1}$, while the usual derivative of x^n is nx^{n-1}. (A q-analogue is an object or result in q-calculus that is analogous to an object or result in calculus.)

Background: The q-binomial coefficients

In the field of q-calculus, many objects other than the derivative also have q-analogues, including binomial coefficients. Recall that the binomial coefficient $\binom{n}{k}$ is the number of ways to choose k objects from a collection of n different objects. A formula for $\binom{n}{k}$ is

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k(k-1)(k-2)\cdots 1}.$$

To change this into a q-binomial coefficient, replace each factor with its q-analogue. So the q-binomial coefficient $\binom{n}{k}_q$ is

$$\binom{n}{k}_q = \frac{[n]_q \cdot [n-1]_q \cdot [n-2]_q \cdots [n-k+1]_q}{[k]_q \cdot [k-1]_q \cdot [k-2]_q \cdot \cdots \cdot [1]_q}.$$

Binomial coefficients satisfy combinatorial identities like Pascal's identity, which states that

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k},$$

and it turns out that q-binomial coefficients satisfy q-analogues of many of these identities. For example, here is the ***q*-Pascal identity**:

$$\binom{n}{k}_q = \binom{n-1}{k-1}_q + q^k \binom{n-1}{k}_q.$$

The s-derivative

Since the q-derivative is linear and satisfies the q-power rule, the q-derivative of a polynomial

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

is

$$a_1 [1]_q + a_2 [2]_q x + \cdots + a_n [n]_q x^{n-1}.$$

What would happen if we generalize the q-derivative by replacing every $[m]_q$ with some other value $s(m)$ that depends on m? Specifically, let s be a sequence with zeroth term $s(0) = 0$. Then, we define the ***s*-derivative** of $f(x)$ to be

$$a_1 s(1) + a_2 s(2) x + \cdots + a_n s(n) x^{n-1}.$$

The s-derivative of x^n is $s(n)x^{n-1}$, a fact called the s-power rule. We say that this is an ***s*-analogue** of the power rule.

The s-binomial coefficients

We can generalize the q-binomial coefficients in a similar way. We start with the definition of $\binom{n}{k}_q$, and replace every $[m]_q$ with $s(m)$, where s is a sequence. The s-binomial coefficient $\binom{n}{k}_s$ is defined to be

$$\binom{n}{k}_s = \frac{s(n)s(n-1)s(n-2)\cdots s(n-k+1)}{s(k)s(k-1)s(k-2)\cdots s(1)}.$$

We proved an analogue of Pascal's identity for the s-binomial coefficients:

$$\binom{n}{k}_s = \binom{n-1}{k-1}_s + \frac{s(n) - s(k)}{s(n-k)} \binom{n-1}{k}_s.$$

Notice that if $(s(n) - s(k))/s(n-k)$ is an integer for all integers $n \geq k \geq 0$, then an induction proof shows that $\binom{n}{k}_s$ is always an integer. This condition on s turns out to imply the existence of s-analogues of several more combinatorial identities.

Definition

We call an integer sequence s a **generalized *n*-series** if it satisfies the following conditions:

1. $s(0) = 0$,
2. $s(n)$ is nonzero for any positive n,
3. $s(n-k)$ divides $s(n) - s(k)$ for all integers $n \geq k \geq 0$.

The term "generalized n-series" comes from the fact that sequences called the ***n*-series of formal group laws** are important examples of generalized n-series.

Other results

We showed that if s is a generalized n-series, then s-analogues of the following results exist:

- the product rule,
- the binomial theorem,
- Vandermonde's identity,
- Lucas's theorem,
- the Poincaré lemma for the algebraic de Rham complex,
- the Cartier isomorphism for the algebraic de Rham complex.

We also studied the asymptotic growth of integer generalized n-series.