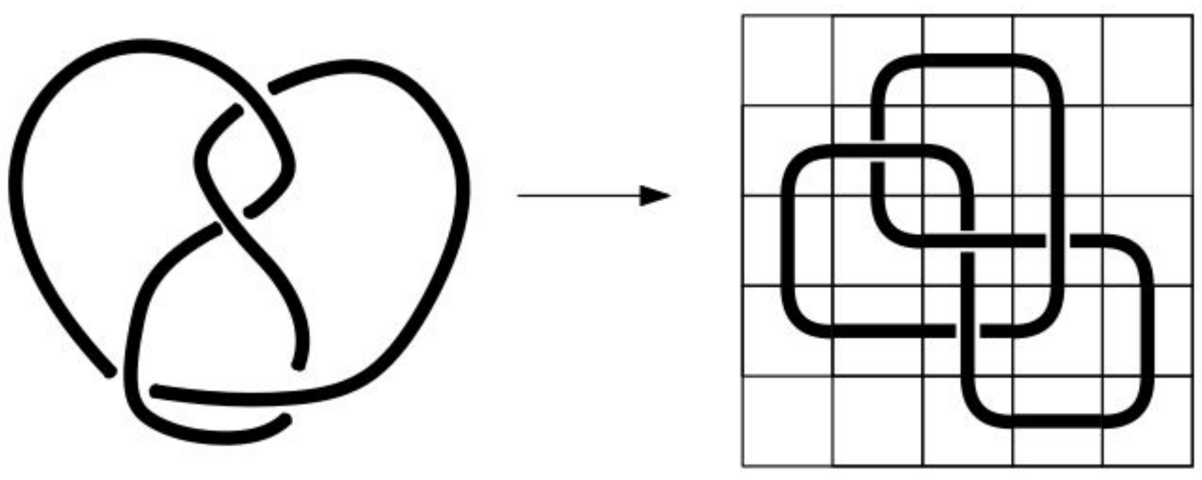
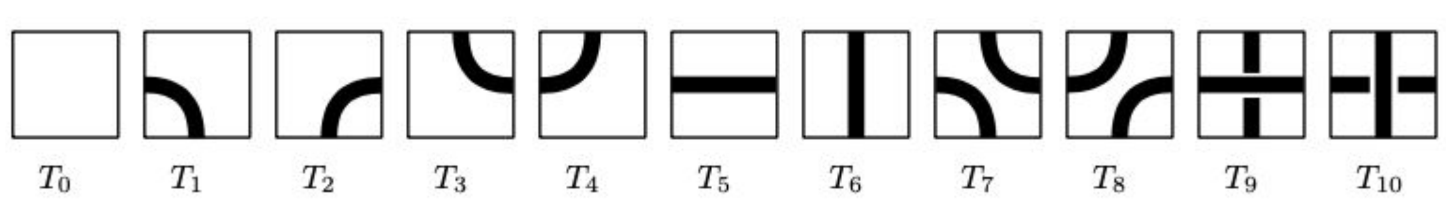
Computing the Mosaic Number of Reduced and Non-Reduced Projections of Knots

Introduction to Knot Mosaics

A knot mosaic is a representation of a knot on an n by n grid. For example, the mosaic representation of the figure eight knot is given below.



The mosaic grids are composed of the 11 tiles shown below, denoted T_0 through T_{10}. Every knot can be represented using only these 11 tiles.

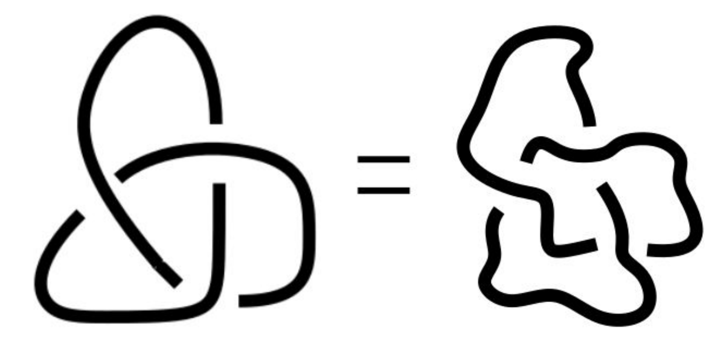


Definition (mosaic number): For a knot K, we define the *mosaic number* m(K) as the smallest integer n such that there exists a projection of K that can be represented on n by n mosaic grid.

Preliminaries

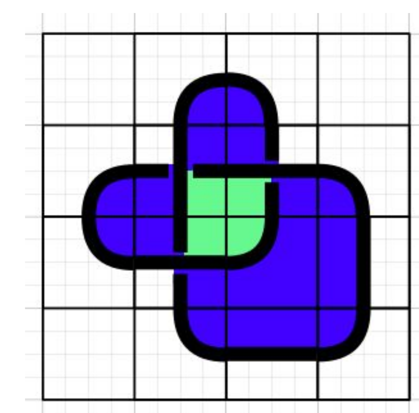
Definition (knot): A *knot* is a closed curve in 3-space that does not intersect itself anywhere. We do not distinguish between the original closed knotted curve and the deformations of that curve through space that do not allow the curve to pass through itself.

Two different projections of the trefoil knot



Definition (crossing number): The *crossing number* of a knot K is the minimal number of crossings of any projection of K. It is denoted c(K).

Definition (n-gon of a knot): An n-gon is a shape found by starting at any crossing tile of the knot, traveling counterclockwise along the arc of that tile and rotating 90° clockwise at every crossing tile reached. If n crossing tiles were reached in the making of this shape, it is an n-gon.



The n-gons in a trefoil with 3-gons in green and 2-gons in blue.

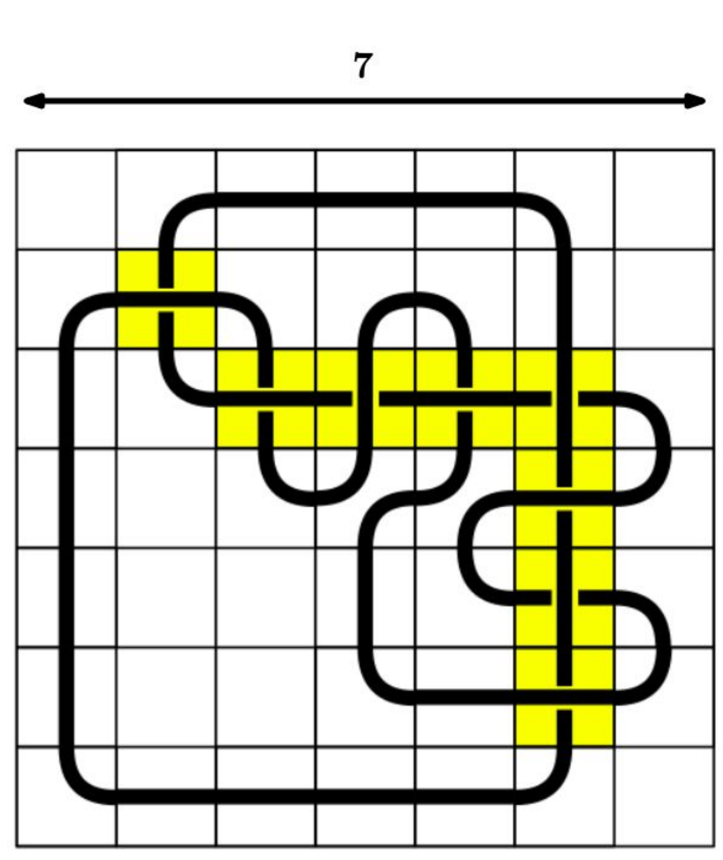
The Case of the 8_3 knot

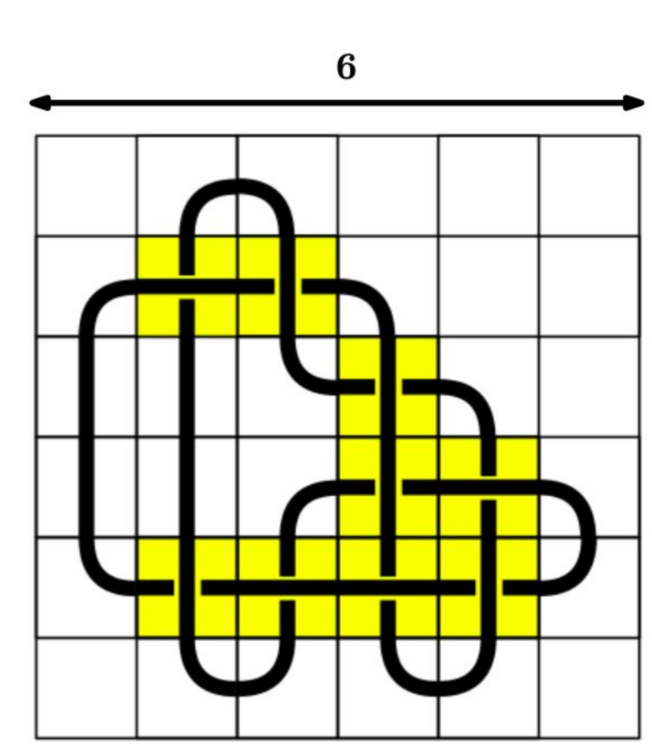
A natural question one can ask is how the knot mosaic number relates to other knot invariants. In particular, the mosaic number can be bounded by the crossing number of a knot K, that is the minimum number of crossings in any projection of K. Intuitively, it seems that the mosaic number of a knot should be realized when the crossing number is realized. If the projection of the knot uses more crossing tiles (tiles T_9 and T_{10}), this may increase the total number of tiles in the knot mosaic. Yet, certain knots realize their mosaic number only in non-reduced projections. It is hard to prove this result for any given knot without enumerating all the possible boards of a certain size.

Question (Lee et al.): Does there exist a representation of 8_3 on a 6 by 6 board with only eight crossing tiles?

Answering a question asked by Lee and others, we look at the case of the 8_3 knot. Mosaic representations of the 8_3 knot in both reduced and non-reduced projections are given below.

The 8_3 knot with 8 crossings on a 7 by 7 mosaic board





The 8_3 knot with 9 crossings on a 6 by 6 mosaic board

Theorem: There is no reduced projection of 8_3 that fits on a 6 by 6 board.

Proof: The proof of this theorem relies on an analysis of the n-gons present in a reduced projection of 8_3. By looking through the different configurations of crossing tiles, we can conclude that no arrangement of 8 crossing tiles on a 6 by 6 board can result in a projection of 8_3.

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